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Inhomogeneous quantum groups related to two-dimensional quantum planes

A Aghamohammadi†§, S Rouhani†‡ and A Shariati†‡

† Department of Physics, Sharif University of Technology, PO Box 9161, Tehran 11365, Iran
‡ Institute for Studies in Theoretical Physics and Mathematics, PO Box 5746, Tehran 19395, Iran and Department of Physics, Alzahra University, Tehran 19834, Iran

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Abstract. We propose a simple and compact method for deriving quantum groups acting on Manin's q- and h-planes. Using this method, we classify all the rigid motions of the two-dimensional quantum planes. Requiring that these inhomogenous transformations preserve differential structure of the quantum plane, we obtain two distinct inhomogenous quantum groups for the q-plane and one previously unknown inhomogenous quantum group for the h-plane, $IGL_{h,h'}(2)$.

1. Introduction

Recently, efforts have been made to construct inhomogeneous quantum groups as inhomogeneous automorphisms of quantum spaces. Two-dimensional quantum planes, the corresponding quantum groups of automorphisms and their differential calculi have been studied by various people [1, 2]. As is well known, there exist only two classes of non-degenerate quantum planes: the q-plane and the h-plane. The former is well studied and the latter has been studied by various authors [1–8].

If $\mathcal{G}$ is a semisimple group that acts on $\mathbb{R}^N$ or $\mathbb{C}^N$, then the corresponding inhomogeneous group $I\mathcal{G}$ is the semidirect product of $\mathcal{G}$ and $\mathbb{R}^N$ or $\mathbb{C}^N$. It is not semisimple since it contains an abelian subgroup isomorphic to $\mathbb{R}^N$ or $\mathbb{C}^N$; its quantization is not straightforward. Since some of the very important groups in physics are inhomogeneous, such as the Poincaré group, it is important to study their quantization.

There is now an expanding literature on the quantization of inhomogeneous groups. Woronowicz [9] used the contraction procedure, whereas Schlieker *et al* [10] introduced a method that uses an R matrix. This method was subsequently put into a more transparent formulation by Castellani [12] and may be called the 'projection' method. Rembielinski [11] used another method to derive the inhomogeneous quantum group that acts on Manin's q-plane.

The purpose of this article is to study the structure of the inhomogeneous quantum groups acting on two-dimensional quantum planes from very basic requirements. We see that the requirement of invariance of the quantum-plane's defining relation leads to an R matrix and that the commutation relations among homogeneous and inhomogeneous sectors are more general than those given by the formalism of Schlieker *et al* [10] and Castellani [12]. Although we have considered the two-dimensional case, the resulting R-matrix formalism

§ E-mail: mohamadi@iream.bitnet

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holds for any dimension. Now, if one demands that the inhomogeneous transformations also preserve the differential structure of the corresponding quantum plane, then the Hopf algebra becomes smaller. For the q-plane, there exist only two distinct differential structures [13] resulting in two different inhomogeneous quantum groups. For the h-plane, there is a known differential structure [6] which leads to $IGL_{h,h'}(2)$.

2. A non-degenerate quantum plane and its inhomogeneous automorphisms

Consider a quantum plane with coordinates $(x, y) = (x_1, x_2)$ and a quadratic relation which we write as

$$C_{ij}x_i x_j = 0 \qquad \text{or} \qquad X^{t}CX = 0 \tag{2.1}$$

where

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.2}$$

C is an invertible matrix and the superscript 't' stands for transposition. We call this quantum plane $\Re^2[C]$.

Now, consider the following inhomogeneous transformation:

$$x \longmapsto x' = ax + by + u \qquad y \longmapsto y' = cx + dy + v. \tag{2.3}$$

Introducing quantum matrix M and translator U, we write this in a more convenient way:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad U = \begin{pmatrix} u \\ v \end{pmatrix} \qquad X \longmapsto X' = M\ X + U. \tag{2.4}$$

Demanding that X' satisfies the same quadratic relation as X, i.e. $X'^{t}CX' = 0$, one gets the following relation:

$$X^{t}M^{t}CMX + (X^{t}M^{t}CU + U^{t}CMX) + U^{t}CU = 0. \tag{2.5}$$

Now, quadratic and linear terms should be zero independently. First, consider the quadratic term

$$X^{t}M^{t}CMX = 0. \tag{2.6}$$

Comparing this with (2.1) leads us to write

$$M^{t}CM = C\sigma \tag{2.7}$$

where σ is a quadratic in the generators a, b, c and d and is called the *'quantum determinant'* of quantum matrix M. The reason for this terminology will be clear soon. The above relations give the definition of σ and three commutation relations among the entries of M. Now, for the linear terms, since x and y are independent variables and commute with the generators a, b, c, d, u and v, we conclude that

$$(M^{t}CU)_i + (U^{t}CM)_i = 0. \tag{2.8}$$

This relation gives us some of the commutation relations between the entries of M and U. Finally, we get

$$U^t C U = 0. \tag{2.9}$$

This simply states that (u, v) have the same commutation relations as (x, y), however note that, while (x, y) commute with entries of the quantum matrix M, (u, v) do not. Using (2.7)–(2.9), one can prove that if (M, U) and (M', U') are two pairs satisfying (2.7)–(2.9), then $(MM', MU' + U)$ is also such a pair provided that all entries of (M, U) mutually commute with all entries of (M', U').

We want to endow our algebra with a coproduct Δ and counity ϵ. We define them as

$$\Delta(M_{ij}) = M_{ik} \otimes M_{kj} \qquad \Delta(U_i) = M_{ik} \otimes U_k + U_i \otimes id \tag{2.10}$$

$$\epsilon(M_{ij}) = \delta_{ij} \qquad \epsilon(U_i) = 0. \tag{2.11}$$

If we put the generators in a 3×3 matrix

$$\mathcal{M} = \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & id \end{pmatrix} \tag{2.12}$$

then expressions (2.10) and (2.11) seem more familiar. Using these definitions and commutation relations (2.7)–(2.9), one can prove that

$$\Delta(\sigma) = \sigma \otimes \sigma \qquad \epsilon(\sigma) = 1 \tag{2.13}$$

$$\sigma(M\ M') = \sigma(M)\ \sigma(M') \qquad \text{if } [M_{ij}, M'_{kl}] = 0. \tag{2.14}$$

These properties justify calling σ the *'quantum determinant'* of matrix M.

For this algebra to be a Hopf algebra, we have to endow it with an antipode. To this end, we assume the existence of an inverse for σ and define

$$\gamma(M) \stackrel{\text{def}}{=} C^{-1} \sigma^{-1} M^t C. \tag{2.15}$$

It is clear from (2.7) that $\gamma(M)M = 1$. But $M\gamma(M) = 1$ only if we have

$$\sigma M \sigma^{-1} C^{-1} M^t C = \sigma \mathbf{1}. \tag{2.16}$$

To complete the commutation relations in such a way that equation (2.16) is satisfied, we consider the following mapping:

$$g \longmapsto \sigma g \sigma^{-1} \qquad g \in \{a, b, c, d, u, v\} \tag{2.17}$$

which is an automorphism of our algebra. This mapping commutes with the following rescaling:

$$(M, U) \longmapsto (M, \lambda U) \qquad \lambda \in \mathbb{C} \tag{2.18}$$

which is also an automorphism of the algebra. Now if we *assume* that (2.17) is a linear transformation in the generator space then we conclude that it does not mix homogeneous and inhomogeneous generators.

A linear transformation of generators that does not mix homogeneous and inhomogeneous generators may be written as

$$M \longmapsto M' = E_{ijmn} e_{ij} M e_{mn} \qquad U \longmapsto U' = FU \tag{2.19}$$

where e_{ij} is a matrix with 1 in the ith row and jth column and zero otherwise; $(e_{ij})_{mn} = \delta_{im}\delta_{jn}$. Now we demand that this linear transformation respect the coproduct Δ and the counity ϵ. After some calculations, one concludes that the most general linear transformation of generators that does not mix homogeneous and inhomogeneous generators and is compatible with the Hopf structure of the algebra $\mathcal{A}^2[C]$ is of the following form for an invertible matrix S and free parameter α:

$$M \longmapsto SMS^{-1} \qquad U \longmapsto \alpha SU. \tag{2.20}$$

We conclude that (2.16) becomes

$$SMS^{-1}C^{-1}M^{\mathrm{t}}C = \sigma \mathbf{1}. \tag{2.21}$$

This gives us the remaining relations among the entries of M and may be written as

$$MDM^{\mathrm{t}} = D\sigma \tag{2.22}$$

where

$$D \stackrel{\mathrm{def}}{=} (CS)^{-1}. \tag{2.23}$$

Now we search for S. First, observe that, since (2.17) is an automorphism of our algebra, it must be true that

$$\sigma M^{\mathrm{t}} \sigma^{-1} C \sigma M \sigma^{-1} = \sigma C \tag{2.24}$$

leading to

$$M^{\mathrm{t}}(S^{\mathrm{t}}CS)M = (S^{\mathrm{t}}CS)\sigma. \tag{2.25}$$

With the same reasoning, we obtain

$$U^{\mathrm{t}}(S^{\mathrm{t}}CS)U = 0 \tag{2.26}$$

$$(S^{-1^{\mathrm{t}}} M^{\mathrm{t}} S^{\mathrm{t}} CSU)_i + (U^{\mathrm{t}} S^{\mathrm{t}} CSMS^{-1})_i = 0. \tag{2.27}$$

Comparing equations (2.25), (2.26) with (2.7), (2.9), we deduce that S must be such that

$$S^{\mathrm{t}}CS = \beta C \tag{2.28}$$

for a free parameter β. Note that S and S^{-1} appear simultaneously in (2.21) so we can set $\beta = 1$.

In summary, given an invertible 2×2 matrix C, one can define a quadratic quantum plane $\mathfrak{R}^2[C]$ by relation (2.1), where D is given by (2.23) and S is a matrix that satisfies

$$S^{\mathrm{t}}CS = C. \tag{2.29}$$

The quantum group whose action on $\mathfrak{R}^2[C]$ is given by (2.3) has the following Hopf structure:
(i) Commutation relations

$$M^{t}CM = C\sigma \qquad MDM^{t} = D\sigma \tag{2.30}$$

$$(M^{t}CU)_i + (U^{t}CM)_i = 0 \tag{2.31}$$

$$\sigma M\sigma^{-1} = SMS^{-1} \qquad \sigma U\sigma^{-1} = \alpha SU. \tag{2.32}$$

(ii) Coproduct

$$\Delta(M_{ij}) = M_{ik} \otimes M_{kj} \tag{2.33}$$

$$\Delta(U_i) = M_{ik} \otimes U_k + U_i \otimes id. \tag{2.34}$$

(iii) Counity

$$\epsilon(M_{ij}) = \delta_{ij} \qquad \epsilon(U_i) = 0. \tag{2.35}$$

(iv) Antipode

$$\gamma(M) = C^{-1}\sigma^{-1}M^{t}C. \tag{2.36}$$

Note that since $(M, U)\cdot(M', U') = (MM', MU'+U)$, we have $(M, U)\cdot(M^{-1}, -M^{-1}U) = (1, 0)$. This means that the antipode of the whole algebra is

$$(M, U) \longmapsto (\gamma(M), -\gamma(M)U). \tag{2.37}$$

The algebra $\mathcal{A}^2[C]$ has the following braid matrix:

$$\mathcal{B}_{ijmn} = \delta_{im}\delta_{jn} + \mu D_{ij}C_{mn} \tag{2.38}$$

where μ is a constant obtained by checking the Yang–Baxter equation for the corresponding R matrix (cf [2])

$$\mu = 1/2\left[\pm\sqrt{\rho^2 - 4} - \rho\right] \qquad \rho \overset{\text{def}}{=} \operatorname{tr}(CD^{t}). \tag{2.39}$$

The commutation relations of the generators may be written as

$$\mathcal{B}_{ijkl}M_{km}M_{ln} = M_{ik}M_{jl}\mathcal{B}_{klmn} \tag{2.40}$$

$$u_i u_j = \mathcal{B}_{ijmn}u_m u_n \tag{2.41}$$

$$M_{im}u_j + u_i M_{jm} = \mathcal{B}_{ijkl}(M_{km}u_l + u_k M_{lm}). \tag{2.42}$$

It can be shown that these commutation relations are consistent with the coproduct (2.10) and counity (2.11). Although we obtained these relations by considering the two-dimensional case (2.40), (2.41) and the relevant Hopf structure, equation (2.33)–(2.37) is valid for any dimension. Note that (2.41) is a more general relation than those given in [10, 12].

Now we look at the effect of a change of coordinates in the quantum plane $\Re^2[C]$. Any invertible 2×2 matrix g defines a change of coordinates

$$g \in GL(2, \mathbb{C}) : X \longmapsto X' = gX. \tag{2.43}$$

It is easy to see that $X'^t C' X' = 0$ if $C' = g^{-1^t} C g^{-1}$. The quantum matrix M' acting on this plane is

$$M' = gMg^{-1} \tag{2.44}$$

while $S' = gSg^{-1}$. This means that $\mathcal{A}^2[C]$ and $\mathcal{A}^2[C']$ are equivalent if $C' = g^{-1^t} C g^{-1}$. So, it is natural to define an equivalence relation

$$\begin{array}{l} \Re^2[C] \sim \Re^2[C'] \\ \mathcal{A}^2[C] \sim \mathcal{A}^2[C'] \end{array} \qquad \text{if } C' = g^{-1^t} C g^{-1} \text{ for some invertible } g. \tag{2.45}$$

C completely specifies a quadratic quantum plane and the corresponding quantum group. Therefore, the set that counts the different quantum planes is $GL(2, \mathbb{C})/(C \sim C')$. It can be shown that

$$GL(2, \mathbb{C})\ /(C \sim C') = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & -h \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}\ q \neq 0 \right\}. \tag{2.46}$$

Note that by rescaling, h can be set equal to one, whenever $h \neq 0$. Therefore, we conclude that there exist only two different quantum planes: the q-plane and the h-plane. Now we derive their inhomogeneous quantum groups.

2.1. q-plane

For this plane, we have

$$C = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}. \tag{2.47}$$

Solving equation (2.29) for S and writing (2.23) for D, one gets

$$S = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \qquad D = \begin{pmatrix} 0 & -q^{-1}p^{-1} \\ p & 0 \end{pmatrix} \tag{2.48}$$

resulting in the following commutation relations:

$$\begin{array}{llll} ac = qca & bd = qdb & [a, d] = qcb - q^{-1}bc & \\ ab = qp^2ba & cd = qp^2dc & cb = p^2bc & \sigma \stackrel{\text{def}}{=} ad - qcb \end{array} \tag{2.49}$$

$$(av + uc) - q(cu + va) = 0 \qquad (bv + ud) - q(du + vb) = 0 \tag{2.50}$$

$$uv = qvu \tag{2.51}$$

$$\sigma u = \alpha p u \sigma \qquad \sigma v = \alpha p^{-1} v \sigma. \tag{2.52}$$

This is Rembielinski's $IGL(2)_{q,s,\mu}(2)$ if we set $p^2 = sq^{-2}$ and $\alpha = \mu^{-1}s^{-1/2}q$ (cf [11]).

2.2. *h-plane*

For this plane, we have

$$C = \begin{pmatrix} 0 & 1 \\ -1 & -h \end{pmatrix}. \tag{2.53}$$

Solving equation (2.29) for S and writing (2.23) for D, one obtains

$$S = \begin{pmatrix} 1 & h' - h \\ 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} -h' & -1 \\ 1 & 0 \end{pmatrix} \tag{2.54}$$

resulting in the following commutation relations:

$$\begin{aligned} &[a, c] = hc^2 \qquad [d, b] = h(\sigma - d^2) \qquad [a, d] = hdc - h'ac \\ &[d, c] = h'c^2 \qquad [b, c] = h'ac + hcd \qquad [b, a] = h'(a^2 - \sigma) \\ &\sigma \stackrel{\text{def}}{=} ad - cb - hcd \end{aligned} \tag{2.55}$$

$$[b, v] + [u, d] = h\{d, v\}[a, v] + [u, c] = h\{c, v\} \tag{2.56}$$

$$uv - vu = hv^2 \tag{2.57}$$

$$\sigma u = \alpha(u + (h' - h)v)\sigma \qquad \sigma v = \alpha v\sigma. \tag{2.58}$$

We call this quantum group $IGL(2)_{h,h',\alpha}$.

3. Differential structures

The algebra $\mathcal{A}^2[C]$ is the quantum symmetry of the quantum plane $\Re^2[C]$. This means that relation (2.1) is preserved under transformation (2.4). We extend the transformation to the whole differential structure as

$$\begin{aligned} x_i &\longmapsto x_i' = M_{ij}x_j + u_i \\ \mathrm{d}x_i &\longmapsto \mathrm{d}x_i' = M_{ij}\,\mathrm{d}x_j \\ \partial_i &\longmapsto \partial_i' = \gamma(M^t)_{ij}\partial_j. \end{aligned} \tag{3.1}$$

We assume that the co-plane, spanned by the partial derivatives ∂_j, also satisfies a quadratic relation

$$\partial_i D'_{ij}\partial_j = 0. \tag{3.2}$$

Requiring that (3.2) be covariant under quantum transformations (3.1) and that M_{ij} satisfy no further relations to those of (2.40), we find that

$$D' = D^{\mathrm{t}} \tag{3.3}$$

where D is given by (2.23). Furthermore, arguments similar to those used in the previous section lead to the existance of the matrix $S_{ij,kl}$ such that

$$\partial_i\partial_j = S_{ij,kl}\partial_k\partial_l \tag{3.4}$$

and

$$\mathcal{S}_{ij,kl} = \delta_{ik}\delta_{jl} + \mu D_{lk} C_{ji} \tag{3.5}$$

with μ given as before. Note that expression (3.5) can be rewritten in terms of a permutation matrix $\mathcal{P}$

$$\mathcal{S} = \mathcal{P}\mathcal{B}\mathcal{P} \tag{3.6}$$

where $\mathcal{B}$ is given by (2.38) [14, 15]. To fully determine the differential structure, we ought to give commutation relations among the coordinates of the plane and the co-plane

$$\partial_j x_i = \delta_{ji} + \mathcal{Q}_{jm,il} x_m \partial_l. \tag{3.7}$$

All the other relations (involving dx^l) will be determined by $\mathcal{Q}_{ij,kl}$, where $\mathcal{Q}$ has to satisfy certain relationships as well as the (Braid) equation [15]

$$(\mathcal{B} - I)(I + \mathcal{Q}) = 0 \tag{3.8}$$

$$\mathcal{B}_{12}\mathcal{Q}_{23}\mathcal{Q}_{12} = \mathcal{Q}_{23}\mathcal{Q}_{12}\mathcal{B}_{23} \tag{3.9}$$

$$\mathcal{Q}_{12}\mathcal{Q}_{23}\mathcal{Q}_{12} = \mathcal{Q}_{23}\mathcal{Q}_{12}\mathcal{Q}_{23} \tag{3.10}$$

where $\mathcal{B}_{12} = \mathcal{B} \otimes 1$, etc. Requiring that (3.7) remain covariant under transformations given by (3.1), we find that

$$M_{nj} M_{mi} \mathcal{Q}_{kl,nm} = \mathcal{Q}_{nm,ji} M_{kn} M_{lm} \tag{3.11}$$

and

$$\mathcal{Q}_{il,jn} M_{ki} u_l = u_j M_{kn}. \tag{3.12}$$

Therefore, if M_{ij} satisfy no further conditions than those of (2.40), we observe that $\mathcal{Q}$ should be proportional to B. The constant of proportionality is fixed by (3.8)

$$\mathcal{Q}_{ij,kl} = \frac{-1}{1+\rho\mu} \mathcal{B}_{ij,kl} \tag{3.13}$$

where ρ, μ and B are given by (2.38) and (2.39). This fixes the differential structure and gives the standard differential structure for the q-plane [14] and the h-plane [6]. Furthermore, we observe that the covariance of the differential structure has produced relations stronger than those of (2.42). We have, thus, reduced $\mathcal{A}^2[C]$ to a smaller algebra $\mathcal{B}^2[C]$, which we believe is the appropriate inhomogeneous quantum group.

3.1. q-plane

The standard differential structure on Manin's q-plane is given by

$$\begin{aligned} & x\,\mathrm{d}x = q^2p^2\,\mathrm{d}x\,x \qquad x\,\mathrm{d}y = (q^2p^2-1)\,\mathrm{d}x\,y + q\,\mathrm{d}y\,x \\ & y\,\mathrm{d}x = qp^2\,\mathrm{d}x\,y \qquad y\,\mathrm{d}y = q^2p^2\,\mathrm{d}y\,y \end{aligned} \tag{3.14}$$

$$\mathrm{d}x\,\mathrm{d}y = -q^{-1}p^{-2}\,\mathrm{d}y\,\mathrm{d}x \qquad \mathrm{d}x^2 = 0 \qquad \mathrm{d}y^2 = 0. \tag{3.15}$$

Equations (3.14) and (3.15) are consistent with the inhomogeneous quantum group $IGL_{q,p}(2)$:

$$\begin{aligned} & ua = q^2p^2au \qquad ub = q^2p^2bu \\ & uc = (q^2p^2-1)av + qcu \qquad ud = (q^2p^2-1)bv + qdu \\ & va = qp^2av \qquad vb = qp^2bv \\ & vc = q^2p^2cv \qquad vd = q^2p^2dv. \end{aligned} \tag{3.16}$$

Rembielinski [11] derived two quantum groups $IGL^{\pm}_{q,s}(2)$. Our result can be transformed to $IGL^{-}_{q,s}(2)$ by setting $s = q^2p^2$. It is interesting to note that $IGL^{+}_{q,s}(2)$ and $IGL^{-}_{q,s}(2)$ are isomorphic to each other. The isomorphism is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in IGL^{-}_{q,s}(2) \longmapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in IGL^{+}_{q^{-1},s^{-1}}(2) \tag{3.17}$$

or in the language of equivalence relation (2.45), there exists a matrix

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.18}$$

that relates these two quantum groups. Also, note that two roots of equation (2.39) result in two equivalent quantum groups, described above.

3.2. h-plane

A differential structure for the h-plane has been obtained in [6].

$$\begin{aligned} & [x,\mathrm{d}x] = h'(\mathrm{d}y\,x + h\,\mathrm{d}y\,y - \mathrm{d}x\,y) \qquad [x,\mathrm{d}y] = h\,\mathrm{d}y\,y \\ & [y,\mathrm{d}x] = -h\,\mathrm{d}y\,y \qquad [y,\mathrm{d}y] = 0 \end{aligned} \tag{3.19}$$

$$\mathrm{d}x\,\mathrm{d}y + \mathrm{d}y\,\mathrm{d}x = 0 \qquad \mathrm{d}y^2 = 0 \qquad \mathrm{d}x^2 = h'\,\mathrm{d}x\,\mathrm{d}y. \tag{3.20}$$

Demanding that transformation (3.1) preserves this differential structure leads to (2.55), (2.57) and the following relations:

$$\begin{aligned} & [u,a] = h'(cu + hcv - av) \qquad [u,b] = h'(du + hdv - bv) \\ & [u,c] = hcv \qquad [u,d] = hdv \\ & [v,a] = -hcv \qquad [v,b] = -hdv \\ & [v,c] = 0 \qquad [v,d] = 0. \end{aligned} \tag{3.21}$$

These complete the commutation relations for $IGL_{h,h'}(2)$.

4. Discussion

In summary, we have shown that the quantum plane, as characterized by matrix C, admits a unique quantum group of rigid transformations $\mathcal{A}^2[C]$, and, further, requiring that the differential structure be covariant leads to a smaller algebra $\mathcal{B}^2[C]$. Thus, a classification of the quantum plane, its rigid motion and differential calculus can be performed in terms of the equivalance classes of C. This leads to only two distinct types of quantum planes; the q-plane and the h-plane, and their respective differential structures.

In the classical limit, translations form an abelian subgroup within the group of inhomogeneous transformations. However, in the quantum case, this is not true. Setting M equal to unity leads to inconsistency among (2.40)–(2.42). The corresponding consistent subgroup results from taking a diagonal M. In the classical limit, this subgroup involves scalings and translations. If we restrict our attention to this limited set of transformations, an alternative quantum group $IRL_{q,s,r}(2)$ exists [11], which is a subgroup of $\mathcal{A}^2[C]$ but not of $\mathcal{B}^2[C]$:

$$ad = da \tag{4.1}$$

$$ua = rau \qquad ud = qdu \tag{4.2}$$

$$av = qva \qquad dv = s^{-1}vd. \tag{4.3}$$

The differential structure is

$$x\,\mathrm{d}x = r\,\mathrm{d}x\,x \qquad x\,\mathrm{d}y = q\,\mathrm{d}y\,x \tag{4.4}$$

$$y\,\mathrm{d}x = q^{-1}\,\mathrm{d}x\,y \qquad y\,\mathrm{d}y = s\,\mathrm{d}y\,y \tag{4.5}$$

$$\mathrm{d}x\,\mathrm{d}y = -q\,\mathrm{d}y\,\mathrm{d}x \qquad \mathrm{d}x^2 = 0 \qquad \mathrm{d}y^2 = 0.$$

In conclusion, we note that this differential structure, which is consistent with $IRL_{q,s,r}(2)$, is the second differential structure given in [13]. An alternative differential structure has become possible because now M satisfies further conditions and has fewer independent elements.

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